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Test Problems for Large-Scale Nonsmooth Minimization*

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Abstract

Many practical optimization problems involve nonsmooth (that is, not necessarily differentiable) functions of hundreds or thousands of variables with various constraints. However, there exist only few large-scale academic test problems for nonsmooth case and there is no established practice for testing solvers for large-scale nonsmooth optimization. For this reason, we now collect the nonsmooth test problems used in our previous numerical experiments and also give some new problems. Namely, we give problems for unconstrained, bound constrained, and inequality constrained nonsmooth minimization.

1 Introduction

Many practical optimization problems involve nonsmooth functions with large amounts of variables (see, e.g., [1, 2, 14]). However, there is no established practice for testing solvers for large-scale nonsmooth optimization and only few large-scale nonsmooth academic test problems exist. In this paper, we give a collection of problems for large-scale nonsmooth minimization. The general formula for these problems is written by

$$\begin{cases} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in G, \end{cases} \tag{1}$$

where the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is supposed to be locally Lipschitz continuous on the feasible region $G \subset \mathbb{R}^n$ and the number of variables n is supposed to be large. Note that no differentiability or convexity assumptions are made.

We shall describe three groups of nonsmooth test problems: unconstrained ($G = \mathbb{R}^n$ in (1), see Section 2), bound constrained ($G = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_i^l \leq x_i \leq x_i^u$ for all $i = 1, \ldots, n\}$ in (1), see Section 3), and inequality constrained ($G = \{\boldsymbol{x} \in \mathbb{R}^n \mid g_j(\boldsymbol{x}) \leq 0$ for all $j = 1, \ldots, p\}$ in (1), see Section 4).

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2 Unconstrained problems.

In this section we present 10 nonsmooth unconstrained ($G = \mathbb{R}^n$ in (1)) minimization problems first introduced in [7]. The problems have been constructed either by chaining and extending small existing nonsmooth problems or by "nonsmoothing" large smooth problems (that is, for example, by replacing the term x_i^2 by $|x_i|$). All these problems can be formulated with any number of variables. We first give the formulation of the objective function f and the starting point $\boldsymbol{x}_1 = (x_1^{(1)}, \dots, x_n^{(1)})^T$ for each problem. Then, we collect some details of the problems as well as the references to the original (small-scale) problems in Table 1.

2.1. Generalization of MAXQ

$$f(\boldsymbol{x}) = \max_{1 \le i \le n} x_i^2.$$

$x_i^{(1)} = i,$ for $i = 1, \dots, n/2$ and
$x_i^{(1)} = -i,$ for $i = n/2 + 1, \dots, n.$

2.2. Generalization of MXHILB

$$f(\boldsymbol{x}) = \max_{1 \le i \le n} \left| \sum_{j=1}^{n} \frac{x_j}{i+j-1} \right|.$$

$x_i^{(1)} = 1.0,$ for all $i = 1, \dots, n.$

2.3. Chained LQ

$$f(\boldsymbol{x}) = \sum_{i=1}^{n-1} \max \left\{ -x_i - x_{i+1}, -x_i - x_{i+1} + (x_i^2 + x_{i+1}^2 - 1) \right\}.$$

$x_i^{(1)} = -0.5,$ for all $i = 1, \dots, n.$

2.4. Chained CB3 I

$$f(\boldsymbol{x}) = \sum_{i=1}^{n-1} \max \left\{ x_i^4 + x_{i+1}^2, (2 - x_i)^2 + (2 - x_{i+1})^2, 2e^{-x_i + x_{i+1}} \right\}.$$

$x_i^{(1)} = 2.0,$ for all $i = 1, \dots, n.$

2.5. Chained CB3 II

$$f(\boldsymbol{x}) = \max \left\{ \sum_{i=1}^{n-1} \left(x_i^4 + x_{i+1}^2 \right), \sum_{i=1}^{n-1} \left((2 - x_i)^2 + (2 - x_{i+1})^2 \right), \sum_{i=1}^{n-1} \left(2e^{-x_i + x_{i+1}} \right) \right\}.$$

$x_i^{(1)} = 2.0,$ for all $i = 1, \dots, n.$

2.6. Number of active faces

$$f(\boldsymbol{x}) = \max_{1\le i\le n} \left\{ g\left(-\sum_{i=1}^{n} x_i\right), g(x_i) \right\},$$

where $g(y) = \ln(|y| + 1)$.

$x_i^{(1)} = 1.0$, for all $i = 1, \ldots, n$.

2.7. Nonsmooth generalization of Brown function 2

$$f(\boldsymbol{x}) = \sum_{i=1}^{n-1} \left(|x_i|^{x_{i+1}^2+1} + |x_{i+1}|^{x_i^2+1} \right).$$

$$\begin{aligned} x_i^{(1)} &= 1.0, && \text{when} \quad \mathrm{mod}\,(i,2) = 0 \quad \text{and} \\ x_i^{(1)} &= -1.0, && \text{when} \quad \mathrm{mod}\,(i,2) = 1, \quad i = 1, \ldots, n. \end{aligned}$$

2.8. Chained Mifflin 2

$$f(\boldsymbol{x}) = \sum_{i=1}^{n-1} \left(-x_i + 2\left(x_i^2 + x_{i+1}^2 - 1\right) + 1.75 \left|x_i^2 + x_{i+1}^2 - 1\right| \right).$$

$x_i^{(1)} = -1.0$, for all $i = 1, \ldots, n$.

2.9. Chained crescent I

$$\begin{aligned} f(\boldsymbol{x}) = \max \Big\{ &\sum_{i=1}^{n-1} \left(x_i^2 + (x_{i+1} - 1)^2 + x_{i+1} - 1\right), \\ &\sum_{i=1}^{n-1} \left(-x_i^2 - (x_{i+1} - 1)^2 + x_{i+1} + 1\right) \Big\}. \end{aligned}$$

$$\begin{aligned} x_i^{(1)} &= 2.0, && \text{when} \quad \mathrm{mod}\,(i,2) = 0 \quad \text{and} \\ x_i^{(1)} &= -1.5, && \text{when} \quad \mathrm{mod}\,(i,2) = 1, \quad i = 1, \ldots, n. \end{aligned}$$

2.10. Chained crescent II

$$\begin{aligned} f(\boldsymbol{x}) = \sum_{i=1}^{n-1} \max \Big\{ &x_i^2 + (x_{i+1} - 1)^2 + x_{i+1} - 1, \\ &-x_i^2 - (x_{i+1} - 1)^2 + x_{i+1} + 1 \Big\}. \end{aligned}$$

$$\begin{aligned} x_i^{(1)} &= 2.0, && \text{when} \quad \mathrm{mod}\,(i,2) = 0 \quad \text{and} \\ x_i^{(1)} &= -1.5, && \text{when} \quad \mathrm{mod}\,(i,2) = 1, \quad i = 1, \ldots, n. \end{aligned}$$

The details of the problems 2.1 – 2.10 are given in Table 1, where p denotes the problem number, $f(\boldsymbol{x}^*)$ is the minimum value of the objective function, and the symbols "$-$" (nonconvex) and "$+$" (convex) denote the convexity of the problems. In addition, the references to the original problems in each case are given in Table 1.

Table 1: Unconstrained problems.

p	$f(\boldsymbol{x}^*)$	Convex	Original problem	Ref.
2.1	0.0	$+$	MAXQ, $n = 20$	[15]
2.2	0.0	$+$	MXHILB, $n = 50$	[10]
2.3	$-(n-1)2^{1/2}$	$+$	LQ, $n = 2$	[16]
2.4	$2(n-1)$	$+$	CB3, $n = 2$	[3]
2.5	$2(n-1)$	$+$	CB3, $n = 2$	[3]
2.6	0.0	$-$	Number of active faces	[5]
2.7	0.0	$-$	Generalization of Brown function	[4]
2.8	varies*	$-$	Mifflin 2, $n = 2$	[6]
2.9	0.0	$-$	Crescent, $n = 2$	[9]
2.10	0.0	$-$	Crescent, $n = 2$	[9]

* $f(\boldsymbol{x}^*) \approx -34.795$ for $n = 50$, $f(\boldsymbol{x}^*) \approx -140.86$ for $n = 200$, and $f(\boldsymbol{x}^*) \approx -706.55$ for $n = 1000$.

3 Bound constrained problems.

In this section we describe 10 nonsmooth bound constrained problems ($G = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_i^l \leq x_i \leq x_i^u$ for all $i = 1, \dots, n\}$ in (1)). Bound constrained problems are easily constructed from the problems given in Section 2 (or in [7]) by inclosing the additional bounds

$$x_i^* + 0.1 \leq x_i \leq x_i^* + 1.1 \qquad \text{for all odd } i.$$

Here $\boldsymbol{x}^*$ denotes the solution point for the unconstrained problem.

If the starting point $\boldsymbol{x}_1 = (x_1^{(1)}, \dots, x_n^{(1)})^T$ given in Section 2 is not feasible, we simply project it to the feasible region (if a strictly feasible starting point is needed an additional safeguard of 0.0001 may be added). The convexity of the bound constrained problems is the same as that of unconstrained problems (see Table 1).

3.1. Bound constrained generalization of MAXQ

$$f(\boldsymbol{x}) = \max_{1 \leq i \leq n} x_i^2.$$

$$0.1 \leq x_i \leq 1.1 \quad \text{when} \quad \mod(i,2) = 0, \quad i = 1, \dots, n.$$

$$\begin{aligned}
x_i^{(1)} &= 1.1, && \text{for } i = 1, \dots, n/2, \text{ when } \mod(i,2) = 0, \\
x_i^{(1)} &= i, && \text{for } i = 1, \dots, n/2, \text{ when } \mod(i,2) = 1, \\
x_i^{(1)} &= 0.1, && \text{for } i = n/2+1, \dots, n, \text{ when } \mod(i,2) = 0, \quad \text{and} \\
x_i^{(1)} &= -i, && \text{for } i = n/2+1, \dots, n, \text{ when } \mod(i,2) = 1.
\end{aligned}$$

3.2. Bound constrained generalization of MXHILB

$f(\boldsymbol{x}) = \max_{1 \le i \le n} \left| \sum_{j=1}^{n} \frac{x_j}{i+j-1} \right|.$

$0.1 \le x_i \le 1.1$ when $\mod(i,2) = 0$, $i = 1, \dots, n$.

$x_i^{(1)} = 1.0$, for all $i = 1, \dots, n$.

3.3. Bound constrained chained LQ

$f(\boldsymbol{x}) = \sum_{i=1}^{n-1} \max \left\{ -x_i - x_{i+1}, -x_i - x_{i+1} + (x_i^2 + x_{i+1}^2 - 1) \right\}.$

$\frac{1}{\sqrt{2}} + 0.1 \le x_i \le \frac{1}{\sqrt{2}} + 1.1$ when $\mod(i,2) = 0$, $i = 1, \dots, n$.

$x_i^{(1)} = \frac{1}{\sqrt{2}} + 0.1$, when $\mod(i,2) = 0$ and
$x_i^{(1)} = -0.5$, when $\mod(i,2) = 1$, $i = 1, \dots, n$.

3.4. Bound constrained chained CB3 I

$f(\boldsymbol{x}) = \sum_{i=1}^{n-1} \max \left\{ x_i^4 + x_{i+1}^2, (2 - x_i)^2 + (2 - x_{i+1})^2, 2e^{-x_i + x_{i+1}} \right\}.$

$1.1 \le x_i \le 2.1$ when $\mod(i,2) = 0$, $i = 1, \dots, n$.

$x_i^{(1)} = 2.0$, for all $i = 1, \dots, n$.

3.5. Bound constrained chained CB3 II

$$f(\boldsymbol{x}) = \max \Big\{ \sum_{i=1}^{n-1} \left(x_i^4 + x_{i+1}^2\right), \sum_{i=1}^{n-1} \left((2 - x_i)^2 + (2 - x_{i+1})^2\right), \sum_{i=1}^{n-1} \left(2e^{-x_i + x_{i+1}}\right) \Big\}.$$

$1.1 \le x_i \le 2.1$ when $\mod(i,2) = 0$, $i = 1, \dots, n$.

$x_i^{(1)} = 2.0$, for all $i = 1, \dots, n$.

3.6. Bound constrained number of active faces

$f(\boldsymbol{x}) = \max_{1 \le i \le n} \left\{ g\left(-\sum_{i=1}^{n} x_i\right), g(x_i) \right\},$

where $g(y) = \ln(|y| + 1)$.

$0.1 \le x_i \le 1.1$ when $\mod(i,2) = 0$, $i = 1, \dots, n$.

$x_i^{(1)} = 1.0$, for all $i = 1, \dots, n$.

3.7. Bound constrained nonsmooth generalization of Brown function 2

$$f(\boldsymbol{x}) = \sum_{i=1}^{n-1} \left(|x_i|^{x_{i+1}^2+1} + |x_{i+1}|^{x_i^2+1} \right).$$

$0.1 \le x_i \le 1.1$ when $\mod(i,2) = 0$, $i = 1, \ldots, n$.

$x_i^{(1)} = 1.0$, when $\mod(i,2) = 0$ and
$x_i^{(1)} = -1.0$, when $\mod(i,2) = 1$, $i = 1, \ldots, n$.

3.8. Bound constrained chained Mifflin 2

$$f(\boldsymbol{x}) = \sum_{i=1}^{n-1} \left(-x_i + 2\left(x_i^2 + x_{i+1}^2 - 1\right) + 1.75\left|x_i^2 + x_{i+1}^2 - 1\right| \right).$$

$0.68 \le x_2 \le 1.68$,
$\frac{1}{\sqrt{2}} + 0.1 \le x_i \le \frac{1}{\sqrt{2}} + 1.1$ when $\mod(i,2) = 0$, $i = 3, \ldots, n-1$,
$0.1 \le x_n \le 1.1$.

$x_2^{(1)} = 0.68$,
$x_i^{(1)} = \frac{1}{\sqrt{2}} + 0.1$, when $\mod(i,2) = 0$, $(i > 2)$, and
$x_i^{(1)} = -1.0$, when $\mod(i,2) = 1$, $i = 1, \ldots, n$.

3.9. Bound constrained chained crescent I

$$f(\boldsymbol{x}) = \max \Big\{ \sum_{i=1}^{n-1} \left(x_i^2 + (x_{i+1}-1)^2 + x_{i+1} - 1\right), \sum_{i=1}^{n-1} \left(-x_i^2 - (x_{i+1}-1)^2 + x_{i+1} + 1\right) \Big\}.$$

$0.1 \le x_i \le 1.1$ when $\mod(i,2) = 0$, $i = 1, \ldots, n$.

$x_i^{(1)} = 1.1$, when $\mod(i,2) = 0$ and
$x_i^{(1)} = -1.5$, when $\mod(i,2) = 1$, $i = 1, \ldots, n$.

3.10. Bound constrained chained crescent II

$$f(\boldsymbol{x}) = \sum_{i=1}^{n-1} \max \Big\{ x_i^2 + (x_{i+1}-1)^2 + x_{i+1} - 1, -x_i^2 - (x_{i+1}-1)^2 + x_{i+1} + 1 \Big\}.$$

$0.1 \le x_i \le 1.1$ when $\mod(i,2) = 0$, $i = 1, \ldots, n$.

$x_i^{(1)} = 1.1$, when $\mod(i,2) = 0$ and
$x_i^{(1)} = -1.5$, when $\mod(i,2) = 1$, $i = 1, \ldots, n$.

4 Inequality constrained problems.

Finally, we describe eight nonlinear or nonsmooth inequality constraints (or constraint combinations). Some of them (constraints 4.1 – 4.5) have been initially given in [8]. The constraints can be combined with the problems given in Section 2 to obtain 80 inequality constrained problems ($G = \{\boldsymbol{x} \in \mathbb{R}^n \mid g_j(\boldsymbol{x}) \leq 0$ for all $j = 1, \ldots, p\}$ in (1)). The constraints are selected such that the original unconstrained minima of problems in Section 2 are not feasible. Note that, due to nonconvexity of the constraints, all the inequality constrained problems formed this way are nonconvex.

The starting points $\boldsymbol{x}_1 = (x_1^{(1)}, \ldots, x_n^{(1)})^T$ for inequality constrained problems are chosen to be strictly feasible. In what follows, the starting points for the problems with constraints are the same as those for problems without constraints (see Section 2) unless stated otherwise.

4.1. Modification of Broyden tridiagonal constraint I
(for original Broyden tridiagonal constraint, see, e.g., [12])

$$g_j(\boldsymbol{x}) = (3.0 - 2.0x_{j+1})x_{j+1} - x_j - 2.0x_{j+2} + 1.0, \qquad j \in [1, n-2],$$

for problems 2.1, 2.2, 2.6, 2.7, 2.9, and 2.10 in Section 2 and

$$g_j(\boldsymbol{x}) = (3.0 - 2.0x_{j+1})x_{j+1} - x_j - 2.0x_{j+2} + 2.5, \qquad j \in [1, n-2],$$

for problems 2.3, 2.4, 2.5, and 2.8 in Section 2.

$x_i^{(1)} = 2.0, \quad i = 1, \ldots, j+2,$ for problems 2.3 and 2.8 in Section 2,
$x_i^{(1)} = 1.0, \quad i = 1, \ldots, j+2,$ for problems 2.9 and 2.10 in Section 2, and
$x_i^{(1)} = -1.0, \quad i \leq j+2$ and $\mathrm{mod}(i, 2) = 0,$ for problem 2.7 in Section 2.

4.2. Modification of Broyden tridiagonal constraint II

$$g_1(\boldsymbol{x}) = \sum_{i=1}^{n-2} \left((3.0 - 2.0x_{i+1})x_{i+1} - x_i - 2.0x_{i+2} + 1.0\right),$$

for problems 2.1, 2.2, 2.6, 2.7, 2.9, and 2.10 in Section 2 and

$$g_1(\boldsymbol{x}) = \sum_{i=1}^{n-2} \left((3.0 - 2.0x_{i+1})x_{i+1} - x_i - 2.0x_{i+2} + 2.5\right),$$

for problems 2.3, 2.4, 2.5, and 2.8 in Section 2.

$x_i^{(1)} = 2.0, \quad i = 1, \ldots, n,$ for problems 2.3 and 2.8 in Section 2.

4.3. Modification of MAD1 I

(for original problem, see, e.g., [13])

$$g_1(\boldsymbol{x}) = \max\left\{x_1^2 + x_2^2 + x_1x_2 - 1.0,\ \sin x_1,\ -\cos x_2\right\},$$
$$g_2(\boldsymbol{x}) = -x_1 - x_2 + 0.5.$$

$x_1^{(1)} = -0.5$ and
$x_2^{(1)} = 1.1$ for all problems in Section 2.

4.4. Modification of MAD1 II

$$g_1(\boldsymbol{x}) = x_1^2 + x_2^2 + x_1x_2 - 1.0,.$$
$$g_2(\boldsymbol{x}) = \sin x_1,$$
$$g_3(\boldsymbol{x}) = -\cos x_2,$$
$$g_4(\boldsymbol{x}) = -x_1 - x_2 + 0.5.$$

$x_1^{(1)} = -0.5$ and
$x_2^{(1)} = 1.1$ for all problems in Section 2.

4.5. Simple modification of MAD1 I

$$g_1(\boldsymbol{x}) = \sum_{i=1}^{n-1}\left(x_i^2 + x_{i+1}^2 + x_ix_{i+1} - 2.0x_i - 2.0x_{i+1} + 1.0\right),$$

for problems 2.1, 2.2, 2.6, 2.7, 2.9, and 2.10 in Section 2 and

$$g_1(\boldsymbol{x}) = \sum_{i=1}^{n-1}\left(x_i^2 + x_{i+1}^2 + x_ix_{i+1} - 1.0\right),$$

for problems 2.3, 2.4, 2.5, and 2.8 in Section 2.

$x_i^{(1)} = 0.5,\quad i = 1,\dots,n,$ for problems 2.1, 2.2, 2.6, 2.7, 2.9, and 2.10 in Section 2 and
$x_i^{(1)} = 0.0,\quad i = 1,\dots,n,$ for problems 2.4, 2.5, and 2.8 in Section 2.

4.6. Simple modification of MAD1 II

$$g_j(\boldsymbol{x}) = x_j^2 + x_{j+1}^2 + x_jx_{j+1} - 2.0x_j - 2.0x_{j+1} + 1.0, \qquad j \in [1, n-1],$$

for problems 2.1, 2.2, 2.6, 2.7, 2.9, and 2.10 in Section 2 and

$$g_j(\boldsymbol{x}) = x_j^2 + x_{j+1}^2 + x_jx_{j+1} - 1.0, \qquad j \in [1, n-1],$$

for problems 2.3, 2.4, 2.5, and 2.8 in Section 2.

$x_i^{(1)} = 0.5,\quad i = 1, \ldots, j+1,$ for problems 2.1, 2.2, 2.6, 2.7, 2.9, and 2.10 in Section 2 and

$x_i^{(1)} = 0.0,\quad i = 1, \ldots, j+1,$ for problems 2.4, 2.5, and 2.8 in Section 2.

4.7. Modification of P20 from UFO collection I
(for original problem, see, e.g., [11])

$$g_j(\boldsymbol{x}) = (3.0 - 0.5x_{j+1})x_{j+1} - x_j - 2.0x_{j+2} + 1.0, \qquad j \in [1, n-2],$$

$x_i^{(1)} = 2.0,\quad i = 1, \ldots, j+2,$ for problems 2.2, 2.3, 2.6, 2.7, 2.9, and 2.10 in Section 2 and

$x_i^{(1)} = -2.0,\quad i = 1, \ldots, j+2,$ for problem 2.8 in Section 2.

4.8. Modification of P20 from UFO collection II

$$g_1(\boldsymbol{x}) = \sum_{i=1}^{n-2} \left((3.0 - 0.50x_{i+1})x_{i+1} - x_i - 2.0x_{i+2} + 1.0\right).$$

$x_i^{(1)} = 2.0,\quad i = 1, \ldots, n,$ for problems 2.2, 2.3, 2.6, 2.7, and 2.8 in Section 2

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References

[1] BELIAKOV, G., MONSALVE TOBON, J. E., AND BAGIROV, A. M. Parallelization of the discrete gradient method of non-smooth optimization and its applications. In *Computational Science — ICCS 2003*, Sloot et. al., Ed., Lecture Notes in Computer Science. Springer Berlin, Heidelberg, 2003, pp. 592–601.

[2] BEN-TAL, A., AND NEMIROVSKI, A. Non-Euclidean restricted memory level method for large-scale convex optimization. *Mathematical Programming 102*, 3 (2005), 407–456.

[3] CHARALAMBOUS, C., AND CONN, A. R. An efficient method to solve the minimax problem directly. *SIAM Journal on Numerical Analysis 15*, 1 (1978), 162–187.

[4] CONN, A. R., GOULD, N. I. M., AND TOINT, P. L. Testing a class of methods for solving minimization problems with simple bounds on the variables. *Mathematics of Computation 50*, 182 (1988), 399–430.

[5] GROTHEY, A. *Decomposition Methods for Nonlinear Nonconvex Optimization Problems*. PhD thesis, University of Edinburgh, 2001.

[6] GUPTA, N. *A Higher than First Order Algorithm for Nonsmooth Constrained Optimization*. PhD thesis, Washington State University, 1985.

[7] HAARALA, M., MIETTINEN, K., AND MÄKELÄ, M. M. New limited memory bundle method for large-scale nonsmooth optimization. *Optimization Methods and Software 19*, 6 (2004), 673–692.

[8] KARMITSA, N., MÄKELÄ, M. M., AND ALI, M. M. Limited memory bundle algorithm for inequality constrained nondifferentiable optimization. Reports of the Department of Mathematical Information Technology, Series B. Scientific Computing, B. 3/2007 University of Jyväskylä, Jyväskylä, 2007.

[9] KIWIEL, K. C. *Methods of Descent for Nondifferentiable Optimization*. Lecture Notes in Mathematics 1133. Springer-Verlag, Berlin, 1985.

[10] KIWIEL, K. C. An ellipsoid trust region bundle method for nonsmooth convex optimization. *SIAM Journal on Control and Optimization 27*, 4 (1989), 737–757.

[11] LUKŠAN, L., TŮMA, M., ŠIŠKA, M., VLČEK, J., AND RAMEŠOVÁ, N. UFO 2002. Interactive system for universal functional optimization. Technical Report 883, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, 2002.

[12] LUKŠAN, L., AND VLČEK, J. Sparse and partially separable test problems for unconstrained and equality constrained optimization. Technical Report 767, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, 1999.

[13] LUKŠAN, L., AND VLČEK, J. Test problems for nonsmooth unconstrained and linearly constrained optimization. Technical Report 798, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, 2000.

[14] MAJAVA, K., HAARALA, N., AND KÄRKKÄINEN, T. Solving variational image denoising problems using limited memory bundle method. In *Proceedings of The 2nd International Conference on Scientific Computing and Partial Differential Equations and The First East Asia SIAM Symposium, Hongkong, December 12-16, 2005.* (to appear, 2006), L. Wenbin, N. Michael, and S. Zhong-Ci, Eds.

[15] SCHRAMM, H. *Eine Kombination von Bundle- und Trust-Region-Verfahren zur Lösung nichtdifferenzierbarer Optimierungsprobleme*. PhD thesis, Bayreuther Mathematische Schriften, No. 30, Universität Bayreuth, 1989.

[16] WOMERSLEY, R. S. *Numerical Methods for Structured Problems in Nonsmooth Optimization*. PhD thesis, Department of Mathematics, University of Dundee, 1981.