Stability analysis for multicriteria discrete optimization problems

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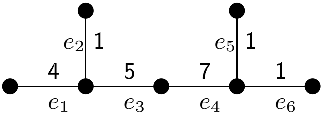
Simple example

Let $G = (V, E)$ be a weighted graph with the edge-set

$$E = \{e_1, e_2, e_3, e_4, e_5, e_6\}.$$

Weights of the edges:
$w(e_1) = 4,\ w(e_2) = 1,\ w(e_3) = 5,\ w(e_4) = 7,\ w(e_5) = 1,\ w(e_6) = 1.$



Let $T = \{t_1, t_2, t_3, t_4\}$ be the set of trajectories, and a weight $w(t)$ be $\sum_{e_i \in t} w(e_i)$.

$$\begin{aligned}
t_1 &= \{e_1, e_2, e_3\} &\Rightarrow\quad & w(t_1) = 4 + 1 + 5 = 10\\
t_2 &= \{e_2, e_4, e_5, e_6\} &\Rightarrow\quad & w(t_2) = 1 + 7 + 1 + 1 = 10\\
t_3 &= \{e_3, e_4\} &\Rightarrow\quad & w(t_3) = 5 + 7 = 12\\
t_4 &= \{e_1, e_2, e_3, e_6\} &\Rightarrow\quad & w(t_4) = 4 + 1 + 5 + 1 = 11
\end{aligned}$$

Optimization problem Z:

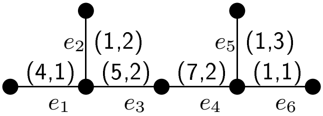
- what is the minimum and the minimizer among the set of trajectories?

It's obvious that

- ✓ t_1 and t_2 are the minimizers in this problem,
- ✓ the minimum is 10.

Upgraded example

Let a weight of each edge be the vector $w = (w_1, w_2)$.
Weights of the edges:
$w(e_1) = (4,1)$, $w(e_2) = (1,2)$, $w(e_3) = (5,2)$,
$w(e_4) = (7,2)$, $w(e_5) = (1,3)$, $w(e_6) = (1,1)$.



A weight $w(t)$ is $\left(\sum\limits_{e_i \in t} w_1(e_i), \sum\limits_{e_i \in t} w_2(e_i)\right)$.

$$
\begin{aligned}
t_1 &= \{e_1, e_2, e_3\} &\Rightarrow\quad & w(t_1) = (10,5)\\
t_2 &= \{e_2, e_4, e_5, e_6\} &\Rightarrow\quad & w(t_2) = (10,8)\\
t_3 &= \{e_3, e_4\} &\Rightarrow\quad & w(t_3) = (12,4)\\
t_4 &= \{e_1, e_2, e_3, e_6\} &\Rightarrow\quad & w(t_4) = (11,6)
\end{aligned}
$$

Optimization problem Z:

- what is the minimum and the minimizer among the set of trajectories?

Pareto efficiency

$$w(t_1) = (10,5),\ w(t_2) = (10,8),\ w(t_3) = (12,4),\ w(t_4) = (11,6)$$

- What is the best trajectory for the problem Z?

Pareto optimality concept: t is an optimal trajectory[1] for the problem Z, if there is no other $t' \in T$ such that t' is better[2] than t.

- ✓ Is t_1 Pareto optimal? $\nexists t : w(t) \lneq w(t_1) \Rightarrow t_1$ is Pareto optimal
- × Is t_2 Pareto optimal?
  $w(t_1) = (10,5) \lneq (10,8) = w(t_2) \Rightarrow t_2$ is not Pareto optimal
- ✓ Is t_3 Pareto optimal? $\nexists t : w(t) \lneq w(t_3) \Rightarrow t_3$ is Pareto optimal
- × Is t_4 Pareto optimal?
  $w(t_1) = (10,5) \lneq (11,6) = w(t_4) \Rightarrow t_4$ is not Pareto optimal

[1] it is called efficient or nondominated or Pareto optimal

[2] it is written $t' \prec t$ and it means $w(t') \lneq w(t)$, i. e. $w(t') \leq w(t)$ & $w(t') \neq w(t)$, i. e. $\forall i\ w_i(t') \leq w_i(t)$ & $\exists k\ w_k(t') < w_k(t)$

n-objective (vector) combinatorial problem

Consider the n-objective (vector) combinatorial problem

$$Z^n(T, f, A): \quad f(t, A) = (f_1(t, A_1), f_2(t, A_2), \ldots, f_n(t, A_n)) \to \min_{t \in T},$$

where A_i is the i-th row of matrix[3] $A = [a_{ij}] \in \mathbf{R}^{n \times m}$, $n \geq 1$, $m \geq 2$, T is a family of nonempty subsets of $N_m = \{1, 2, \ldots, m\}$ (called trajectories), i.e. $T \subseteq 2^{N_m} \setminus \{\emptyset\}$, $|T| \geq 2$.

[3] Actually A is weight matrix of graph edges, so a_{ij} is weight of j-th edge by i-th criterion

1-objective (scalar) combinatorial problem

It is easy to see that many extreme problems on graphs such as

1. the traveling salesman problem,
2. the assignment problem,
3. the shortest path problem,
4. the minimum spanning tree problem,
5. etc...

are included in the similar scheme of scalar[4] combinatorial problems (with linear, minimax and other criteria types).

[4] $n = 1$

Pareto set

Definition 1

A trajectory $t \in T$ is called an efficient solution of problem $Z^n(T, f, A)$, if no other trajectory dominates it (in the sense of Pareto), i.e. there does not exist t' such that $t \underset{f,A}{\succ} t'$, where

$$t \underset{f,A}{\succ} t' \quad \overset{def}{\Longleftrightarrow} \quad \forall i \in N_n \quad \left(f_i(t, A_i) \geq f_i(t', A_i)\right) \quad \& \\ \exists k \in N_n \quad \left(f_k(t, A_k) > f_k(t', A_k)\right).$$

The set of efficient trajectories of problem $Z^n(T, f, A)$ is denoted by $P^n(f, A)$ and called the Pareto set (Pareto, 1896).
So we can write

$$P^n(f, A) = \left\{ t \in T : \ \nexists t' \in T \setminus \{t\} \quad \left(t \underset{f,A}{\succ} t' \right) \right\}.$$

Smale and Slater sets

Let us put into consideration the Smale set[5] (Smale, 1974) and the Slater set[6] (Slater, 1950) correspondingly:

$$Sm^n(f,A) = \left\{ t \in T : \ \nexists t' \in T \setminus \{t\} \quad \left(t \underset{f,A}{\geqslant} t' \right) \right\},$$

$$Sl^n(f,A) = \left\{ t \in T : \ \nexists t' \in T \setminus \{t\} \quad \left(t \underset{f,A}{>} t' \right) \right\},$$

where

$$t \underset{f,A}{\geqslant} t' \quad \overset{\text{def}}{\Longleftrightarrow} \quad \forall i \in N_n \quad \left(f_i(t, A_i) \geq f_i(t', A_i) \right),$$

$$t \underset{f,A}{>} t' \quad \overset{\text{def}}{\Longleftrightarrow} \quad \forall i \in N_n \quad \left(f_i(t, A_i) > f_i(t', A_i) \right).$$

It is evident that

$$\emptyset \subseteq Sm^n(f,A) \subseteq P^n(f,A) \subseteq Sl^n(f,A)$$

and $P^n(f,A)$ cannot be empty.

[5] it is also called the set of strictly efficient trajectories
[6] it is also called the set of weakly efficient trajectories

The perturbation of problem parameters

The perturbation of problem parameters is understood as an arbitrary change of elements of matrix A.

A perturbation is modeled by adding perturbing matrix $A' \in \mathbf{R}^{n\times m}$ to matrix A. Thus a perturbed problem is formulated as follows:

$$Z^n(T, f, A + A') : \qquad f(t, A + A') \to \min_{t\in T}$$

or

$$Z^n(T, f, B) : \qquad f(t, B) \to \min_{t\in T},$$

where $B = A + A'$.

Stability in continuous and discrete problems

Stability is

in continuous problem: *continuous dependence* of solutions *on initial data* of the problem,

in discrete problem: a certain *preassigned property of invariance to external influence on initial data* of the problem.

Stability and quasi-stability

Definition 2

The problem $Z^n(T, f, A)$ *is called* *stable* *if*

$$\exists \varepsilon > 0 \;\; \forall B \in \Omega_\varepsilon(A) \;\; \left(P^n(f, B) \subseteq P^n(f, A)\right).$$

Definition 3

The problem $Z^n(T, f, A)$ *is called* *quasi-stable* *if*

$$\exists \varepsilon > 0 \;\; \forall B \in \Omega_\varepsilon(A) \;\; \left(P^n(f, A) \subseteq P^n(f, B)\right).$$

Here
$\Omega_\varepsilon(A) = \left\{B \in \mathbf{R}^{n \times m} : \;\|B - A\| < \varepsilon\right\}$ is the ε-neighborhood of a point A,
$\|\cdot\|$ is an arbitrary norm on a space $\mathbf{R}^{n \times m}$.

Two approaches to stability analysis

There are two major directions of investigation of stability analysis for discrete optimization problems:

quantitative
: to derive quantitative bounds for feasible initial data changes preserving some preassigned properties of optimal solutions,

qualitative
: to obtain conditions under which the set of optimal solutions of the problem possesses a certain preassigned property of invariance to external influence on initial data of the problem.

Some more definitions and notations

Definition 4

We call

$$N_i(t, f_i, A_i) = \Big\{ j \in N_m : \ \forall \varepsilon > 0 \ \exists \delta \in (-\varepsilon, \varepsilon) \quad \big(f_i(t, A_i) \neq f_i(t, A_i + \delta I_j)\big) \Big\}$$

the set of sensitive elements.

Here I_j is j-th row of identity matrix I of size $m \times m$.

Put also

$$N_i^+(t, f_i, A_i) = \Big\{ j \in N_m : \ \forall \varepsilon > 0 \ \exists \delta \in (-\varepsilon, \varepsilon) \quad \big(f_i(t, A_i) < f_i(t, A_i + \delta I_j)\big) \Big\},$$

$$N_i^-(t, f_i, A_i) = \Big\{ j \in N_m : \ \forall \varepsilon > 0 \ \exists \delta \in (-\varepsilon, \varepsilon) \quad \big(f_i(t, A_i) > f_i(t, A_i + \delta I_j)\big) \Big\},$$

$$g_i(t, t', A_i) = f_i(t, A_i) - f_i(t', A_i).$$

α-regularity

Definition 5

We call $f_i(t, B_i)$ α-regular at the point A_i, if there exists $\varepsilon = \varepsilon(i, t) > 0$ such that

(α.1) *on the set $\Omega_\varepsilon(A_i)$ the function $f_i(t, B_i)$ is non-decreasing w.r.t. b_{ij}, $j \in N_i^+(t, f_i, A_i)$, and increasing w.r.t. b_{ij}, $j \in N_i^-(t, f_i, A_i)$;*

(α.2) *for each trajectory $t' \in T$ the function $g_i(t, t', B_i)$ is permanent on $\Omega_\varepsilon(A_i)$ w.r.t. b_{ij}, $j \in N_i^-(t, f_i, A_i) \cap N_i^-(t', f_i, A_i)$;*

(α.3) *for any $B_i \in \Omega_\varepsilon(A_i)$ the inclusion $N_i(t, f_i, B_i) \subseteq N_i(t, f_i, A_i)$ holds.*

Definition 6

We call $f(t, B)$ α-regular at the point A, if $f_i(t, B_i)$ is α-regular at the point A_i for all $i \in N_n$.

The necessary condition for stability

We put

$$W(t, f, A) = \left\{ t' \in P^n(f, A) : \ t \underset{f,A}{\geqslant} t' \right\},$$

$$t \underset{f,A}{\vdash} t' \quad \overset{\text{def}}{\Longleftrightarrow} \quad \forall i \in N_n \quad \big(g_i(t, t', A_i) = 0 \ \Rightarrow$$

$$N_i^+(t, f_i, A_i) \supseteq N_i^+(t', f_i, A_i) \quad \& \quad N_i^-(t', f_i, A_i) \supseteq N_i^-(t, f_i, A_i) \big).$$

Theorem 1.1

Let the function $f = f(t, B)$ be α-regular at the point A for each $t \in Sl^n(f, A)$. If the problem $Z^n(T, f, A)$, $n \geq 1$, is stable, then

$$\forall t \in Sl^n(f, A) \ \exists t' \in W(t, f, A) \quad \left(t \underset{f,A}{\vdash} t' \right).$$

β-regularity

Definition 7

We call $f_i(t, B_i)$ β-regular at the point A_i, if there exists $\varepsilon = \varepsilon(i, t) > 0$ such that

$(\beta.1)$ $f_i(t, B_i)$ *is continuous on the set* $B_i \in \Omega_\varepsilon(A_i)$ *w.r.t.* b_{ij}, $j \in N_m$;

$(\beta.2)$ $g_i(t, t', B_i) \geq 0$ *at any* $B_i \in \Omega_\varepsilon(A_i)$ *for each* t' *such that* $N_i^+(t, f_i, A_i) \supseteq N_i^+(t', f_i, A_i)$ *and* $N_i^-(t', f_i, A_i) \supseteq N_i^-(t, f_i, A_i)$.

Definition 8

We call $f(t, B)$ β-regular at the point A, if there $f_i(t, B_i)$ is β-regular on the point A_i for all $i \in N_n$.

The sufficient condition for stability

Theorem 1.2

Let the function $f = f(t, B)$ be β-regular at the point A for each $t \in T$. If the formula

$$\forall t \in Sl^n(f, A)\ \exists t' \in W(t, f, A) \quad \left(t \underset{f,A}{\vdash} t'\right)$$

is valid, then the problem $Z^n(T, f, A)$, $n \geq 1$, is stable.

Examples of α- and β-regular functions

$$f_i(t, A_i) = \sum_{j \in t} a_{ij}, \qquad \text{(Sum)}$$

$$f_i(t, A_i) = \max_{j \in t} a_{ij}, \qquad \text{(Max)}$$

$$f_i(t, A_i) = \min_{j \in t} a_{ij}, \qquad \text{(Min)}$$

$$f_i(t, A_i) = \sqrt[p]{\sum_{j \in t} a_{ij}^p}, \quad p > 0, \qquad \text{(p-th Root)}$$

$$f_i(t, A_i) = \sqrt[p]{\sum_{j \in t} |a_{ij}|^p}, \quad p > 0. \qquad \text{(p-th Root Mod)}$$

- Linear combination each of this functions is β-regular for all $A \in \mathbf{R}^{n \times m}$ and $t \in T$.
- Linear combination of Sum- and Max-functions is α-regular for all $A \in \mathbf{R}^{n \times m}$ and $t \in T$.

Corollaries

Corollary 1.1

Let $f(t, B)$ be continuous and α-regular at every point $B \in \mathbf{R}^{n\times m}$ for any $t \in T$. Let

$$\forall B_i \in \mathbf{R}^m \quad \forall t \in T \qquad \left(N_i^+(t, f_i, B_i) = N_i^-(t, f_i, B_i) = t\right)$$

for all $i \in N_n$. Than $Z^n(T, f, A)$, $n \geq 1$, is stable if and only if

$$P^n(f, A) = Sl^n(f, A).$$

The following known[7,8] statement obviously follows from Corollary 1.1.

Corollary 1.2

The problem $Z^n(T, f, A)$, $n \geq 1$, with linear partial criteria ($f_i(t, A_i)$, $i \in N_n$, are Sum-functions) is stable if and only if

$$P^n(f, A) = Sl^n(f, A).$$

[7] Girlikh E., Kovalev M. M., Kravtsov M. K. Stability, pseudostability, and quasistability of a multicriterion problem on a system of subsets // Cybernetics and Systems Analysis. 1999. V. 35, N 5. P. 777–788.

[8] Sergienko I. V., Shilo V. P. Discrete Optimization Problems. 2003. Naukova dumka, Kiev.

Let

$$N_i^+(t, A_i) = \text{Argmax}\{a_{ij} : \ j \in t\}.$$

The following known[9] statement is valid.

Corollary 1.3

The problem $Z^n(T, f, A)$, $n \geq 1$, with minmax partial criteria ($f_i(t, A_i)$, $i \in N_n$, are Max-functions) is stable if and only if

$$\forall t \in Sl^n(f, A) \ \exists t' \in P^n(f, A) \ \ \forall i \in N_n$$
$$(g_i(t, t', A_i) = 0 \quad \Rightarrow \quad N_i^+(t, A_i) \supseteq N_i^+(t', A_i)).$$

[9] Emelichev V. A., Kuzmin K. G. Stability criteria in vector combinatorial bottleneck problems in terms of binary relations // Cybernetics and Systems Analysis. 2008. V. 44, N 3. P. 397–404.

Let

$$N_i^-(t, A_i) = \text{Argmin}\{a_{ij} : \ j \in t\}.$$

The following known[10][11] statement is valid.

Corollary 1.4

If

$$\forall t \in Sl^n(f, A) \ \exists t' \in P^n(f, A) \ \ \forall i \in N_n$$
$$(g_i(t, t', A_i) = 0 \ \ \Rightarrow \ \ N_i^-(t', A_i) \supseteq N_i^-(t, A_i)).$$

then the problem $Z^n(T, f, A)$, $n \geq 1$, with minmin partial criteria ($f_i(t, A_i)$, $i \in N_n$, are Min-functions) is stable.

[10] Emelichev V. A., Kuzmin K. G. On stability of a vector combinatorial problem with MINMIN criteria // Discrete Mathematics and Applications. 2008. V. 18, N 6. P. 557–562.

[11] Emelichev V. A., Karelkina O. V., Kuzmin K. G. Qualitative stability analysis of multicriteria combinatorial minimin problems // Control and Cybernetics. 2012. V. 41, N 1. P. 57–79.

The necessary condition for quasi-stability

We put

$$t \underset{f,A}{\sim} t' \quad \overset{\text{def}}{\Longleftrightarrow} \quad f(t,A) = f(t',A) \;\Rightarrow\; \forall i \in N_n \;\; \big(N_i(t,f_i,A_i) = N_i(t',f_i,A_i)\big).$$

Theorem 2.1

If the problem $Z^n(T,f,A)$, $n \geq 1$, is quasi-stable, then

$$\forall t,t' \in P^n(f,A) \qquad \left(t \underset{f,A}{\sim} t'\right).$$

γ-regularity

Definition 9

We call $f_i(t, B_i)$ γ-regular at the point A_i, if there exists $\varepsilon = \varepsilon(i, t) > 0$ such that

(γ.1) $f_i(t, B_i)$ *is continuous on* $B_i \in \Omega_\varepsilon(A_i)$ *w.r.t.* b_{ij}, $j \in N_m$;

(γ.2) $g_i(t, t', B_i)$ *is permanent on* $B_i \in \Omega_\varepsilon(A_i)$ *for any* t' *such that* $N_i(t', f_i, A_i) = N_i(t, f_i, A_i)$.

Definition 10

We call $f(t, B)$ γ-regular at the point A, if there $f_i(t, B_i)$ is γ-regular on the point A_i for all $i \in N_n$.

The sufficient condition for quasi-stability

Theorem 2.2

Let the function $f = f(t, B)$ *be* γ*-regular at the point* A *for each* $t \in T$*. If the formula*

$$\forall t, t' \in P^n(f, A) \qquad \left(t \underset{f,A}{\sim} t'\right)$$

is valid, then the problem $Z^n(T, f, A)$, $n \geq 1$, *is quasi-stable.*

Examples of regular γ-functions

$$f_i(t, A_i) = \max_{j \in t} a_{ij}, \qquad \text{(Max)}$$

$$f_i(t, A_i) = \min_{j \in t} a_{ij}, \qquad \text{(Min)}$$

$$f_i(t, A_i) = \max_{j \in t} |a_{ij}|, \qquad \text{(Max Mod)}$$

$$f_i(t, A_i) = \min_{j \in t} |a_{ij}|, \qquad \text{(Min Mod)}$$

$$f_i(t, A_i) = \sqrt[p]{\sum_{j \in t} a_{ij}^p}, \qquad p > 0, \qquad \text{(p-th Root)}$$

$$f_i(t, A_i) = \sqrt[p]{\sum_{j \in t} |a_{ij}|^p}, \qquad p > 0. \qquad \text{(p-th Root Mod)}$$

- Linear combination each of these functions is γ-regular for every $A \in \mathbf{R}^{n \times m}$ and $t \in T$.

Corollaries

Corollary 2.1

Let $f(t, B)$ *be* γ*-regular at every point* $B \in \mathbf{R}^{n\times m}$ *for any* $t \in T$*. Let*

$$\exists k \in N_n \quad \forall B_k \in \mathbf{R}^m \quad \forall t \in T \qquad (N_k(t, f_k, B_k) = t)\,.$$

Than $Z^n(T, f, A)$, $n \geq 1$, *is quasi-stable if and only if*

$$P^n(f, A) = Sm^n(f, A).$$

Corollaries

Corollary 2.2

The problem $Z^n(T, f, A)$, $n \geq 1$, with p-th Root and p-th Root Mod partial criteria is quasi-stable if and only if

$$P^n(f, A) = Sm^n(f, A).$$

The following known[12] statement obviously follows from Corollary 2.2.

Corollary 2.3

The problem $Z^n(T, f, A)$, $n \geq 1$, with linear partial criteria is quasi-stable if and only if

$$P^n(f, A) = Sm^n(f, A).$$

[12] Girlikh E., Kovalev M. M., Kravtsov M. K. Stability, pseudostability, and quasistability of a multicriterion problem on a system of subsets // Cybernetics and Systems Analysis. 1999. V. 35, N 5. P. 777–788.

Corollaries

The following known[13] statements are valid.

Corollary 2.4

The problem $Z^n(T, f, A)$, $n \geq 1$, *with minmax partial criteria is quasi-stable if and only if*

$$f(t, A) = f(t', A) \quad \Rightarrow \quad \forall i \in N_n \quad \left(N_i^+(t, A_i) = N_i^+(t', A_i)\right).$$

Corollary 2.5

The problem $Z^n(T, f, A)$, $n \geq 1$, *with minmin partial criteria is quasi-stable if and only if*

$$f(t, A) = f(t', A) \quad \Rightarrow \quad \forall i \in N_n \quad \left(N_i^-(t, A_i) = N_i^-(t', A_i)\right).$$

[13] Emelichev V. A., Stepanishina Yu. V. Quasistability of a vector nonlinear trajectory problem with the Pareto optimality principle // Russian Mathematics (Izvestiya VUZ. Matematika). 2000. V. 44. N 12. P. 25–30.

Stability and quasi-stability conditions

Stability conditions

Necessary	Sufficient
f is α-regular	f is β-regular

$$\forall t \in Sl^n(f,A) \quad \exists t' \in W(t,f,A) \quad \forall i \in N_n$$

$$f_i(t,A_i) = f_i(t',A_i) \Rightarrow$$

$$N_i^+(t,f_i,A_i) \supseteq N_i^+(t',f_i,A_i) \;\; \& \;\; N_i^-(t',f_i,A_i) \supseteq N_i^-(t,f_i,A_i)$$

Quasi-stability conditions

Necessary	Sufficient
	f is γ-regular

$$\forall t,t' \in P^n(f,A)$$

$$f(t,A) = f(t',A) \;\; \Rightarrow \;\; \forall i \in N_n \quad \big(N_i(t,f_i,A_i) = N_i(t',f_i,A_i)\big)$$

Thank you for your interest!