Markowitz's Investment Problem under Multicriteria, Uncertainty and Risk

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The problem

$$Z^s(R): \quad f(x) = (f_1(x), \ldots, f_s(x)) \to \min_{x \in X}, \quad s \geq 1,$$

of finding Pareto set

$$P^s(R) = \{x \in X : \nexists x' \in X \ \ (f(x) \geq f(x') \ \& \ f(x) \neq f(x'))\}.$$

Multicriteria investment Boolean problem with Savage's minimax criteria:

$$f_k(x) = \max_{i \in N_m} \sum_{j \in N_n} r_{ijk} x_j \to \min_{x \in X}, \qquad k \in N_s.$$

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$\sum_{j \in N_n} r_{ijk} x_j^0$ – the risk of type k which an investor takes, investing in portfolio x^0 in the case when the market is in state i.

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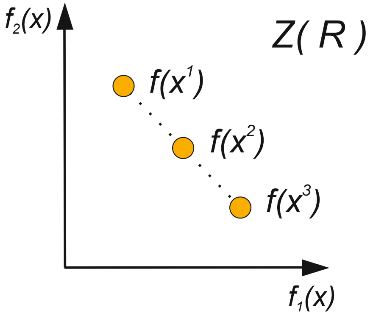

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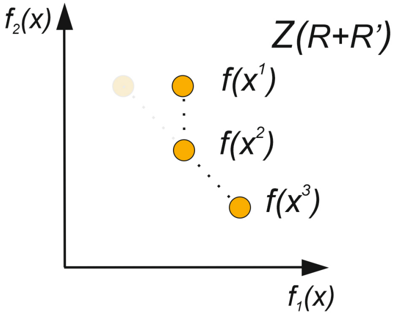

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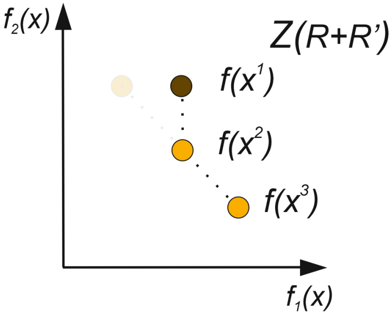

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The stability radius of a Pareto-optimal portfolio x^0 of the problem $Z^s(R)$:

$$\rho_p^{s,m}(x^0, R) = \begin{cases} \sup\ \Xi, & \text{if } \ \Xi \neq \emptyset, \\ 0, & \text{if } \ \Xi = \emptyset, \end{cases}$$

where

$$\Xi = \{\varepsilon > 0 : \ \forall R' \in \Omega(\varepsilon) \quad (x^0 \in P^s(R + R'))\},$$

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$$\|R'\|_{ppp} = \left\| \left(\|R'_1\|_{pp}, \|R'_2\|_{pp}, \dots, \|R'_s\|_{pp} \right) \right\|_p -$$

the norm of the matrix,

$$\|R_k\|_{pp} = \|(\|R_{1k}\|_p, \|R_{2k}\|_p, \dots, \|R_{mk}\|_p)\|_p, \quad k \in N_s,$$

$$\|a\|_p = \begin{cases} \left(\sum_{k\in N_s} |a_k|^p\right)^{1/p}, & \text{if } \ 1 \leq p < \infty, \\ \\ \max\{|a_k| : \ k \in N_s\}, & \text{if } \ p = \infty, \end{cases}$$

$$a = (a_1, a_2, \dots, a_s) \in \mathbb{R}^s.$$

Theorem. *Let*

$$\varphi(x^0) = \min_{x \in X \setminus \{x^0\}} \frac{\|[f(x) - f(x^0)]^+\|_p}{\|x \bullet x^0\|_q},$$

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then for $1 \leq p \leq \infty$, s, $m \in \mathbb{N}$,

$$\varphi(x^0) \leq \rho_p^{s,m}(x^0, R) \leq m^{1/p} \psi(x^0).$$

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Here

$$x \bullet x^0 = (x_1, x_2, \dots, x_n, x_1^0, x_2^0, \dots, x_n^0),$$

$$[a]^+ = (a_1^+, a_2^+, \dots, a_s^+), \quad a_k^+ = \max\{0, a_k\}, \quad k \in N_s,$$

$$1/p + 1/q = 1.$$

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For $1 \le p \le \infty$

$$\varphi(x^1) = 2^{-2+3/p},$$

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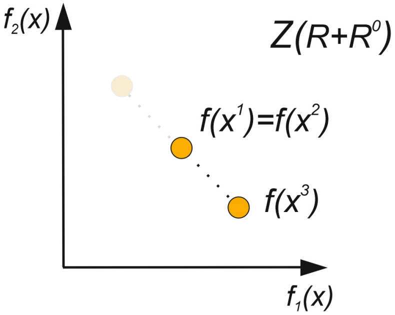

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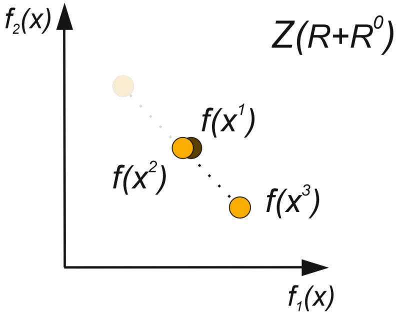

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Corollary 2 [1]. *For* $1 \leq p \leq \infty$, $m = 1$

$$\rho_p^{s,1}(x^0, R) = \psi(x^0) = \min_{x \in X \setminus \{x^0\}} \frac{\|[R(x - x^0)]^+\|_p}{\|x - x^0\|_q}.$$

[1] Emelichev, V.A., Kuz'min, K.G.: A general approach to studying the stability of a Pareto optimal solution of a vector integer linear programming problem. *Discrete Mathematics and Applications* 17, 349–354 (2007)

Thank you for your attention!